Accurate Long-Term Deflection Prediction in Flat Slabs Using Linear Elastic Global Analysis

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Synopsis: With long-term deflection being a critical design criterion, accurate deflection prediction is an important aspect of slab design. The many factors that influence long-term deflections and their interdependence as well as variability in material strengths and characteristics make this a challenging problem. Most building codes and standards provide methods and formulae to account for these factors, although the predictions are sometimes crude. In this paper an approach is presented that comprehensively considers the factors commonly considered important in the calculation of long-term deflections – cracking, tension stiffening, creep, and shrinkage. The load history effects on the applicable factors are also considered. After detailed non-linear calculations are performed on the cross sections, an approach is discussed that uses the cross section results in conjunction with a linear elastic global analysis to calculate accurate two-way slab deflections.

Keywords: concrete, cracking, tension stiffening, strain, deflection, creep, shrinkage, flat slabs

1. Introduction

Due to the use of higher strength materials and more aggressive limit state design, the accurate calculation of deflections has become an important consideration in the design of concrete structures. In interior locations excessive deflections can cause damage to partitions, cladding, mechanical or plumbing. In open structures excessive deflections can cause ponding of water. Proper consideration of deflections must include a number of factors, many of which are time dependent. Some common factors that need to be considered are cracking, tension stiffening, creep, and shrinkage. Load history is also important, as sustained loads on the structure can vary and it is common for the structure to experience loads approaching the full design loads during construction.

In traditional frame structures deflections have been calculated in a number of ways. One traditional approach is to calculate the curvature at stations along the member, then integrate over the length of the member to solve for the deflected shape. Another approach is to equate the work done by the external loads to the internal strain energy in the member. In frame structures with discrete members calculation of deflections by these approaches is fairly straightforward. However, in continuum structures such as two-way slabs these methods can be more difficult to apply. In these cases a linear elastic finite element solution can be employed to calculate deflection contours, however accounting for the inelastic behaviour of concrete and the common long-term effects can be difficult and computationally expensive.

The approach presented in this paper will perform detailed calculations on discrete cross sections then use the calculated cross section deformations to modify the linear elastic global analysis to calculate deflections on a continuum structure such as a flat plate or flat slab. The primary benefit of this approach is that the details of the complex time dependent behaviours are isolated to calculations on the cross section.

2. Calculations on the Cross Section

2.1 General Cross Section Behaviour

The basis of the presented approach is to perform detailed time-dependent curvature calculations on cross sections in order to determine long-term deformations. Curvatures will be calculated for cracked and uncracked states and will include the most significant long-term effects.

One common approach to calculating cross section curvatures is to calculate cracked or uncracked transformed section properties, then use standard mechanics of materials equations to determine the curvature using the calculated properties. Ghali et al. (1) describe a method to apply the long-term effects to such an approach. However, this approach implies linear elastic behaviour for all cross section materials. Additionally, as creep and shrinkage take place the neutral axis depth tends to shift as a function of time. This would mean that at the end of a time period the initial calculated cross section properties might no longer be valid, and thus require iterative checks. This approach is well suited for hand calculations.

A more accurate approach is to employ a stress-strain curve for each material and integrate the resultant material stresses over the cross section to calculate a force resultant. The assumptions that are typically made in this analysis are that plane sections remain plane, and that, in cross sections considered to be cracked, concrete does not carry any tensile stress. For uncracked sections the assumption is made that the concrete stress-strain curve in tension mirrors the compression behaviour.

In solving for the unknown cross section curvatures, a cross section force will generally be given. These cross section forces are normally calculated during the global analysis. In a continuum structure they can be calculated in a number of ways including integrating element stresses or calculated from equilibrium with nodal loads. Using iterative analysis on a computer, the cross section curvatures are adjusted until the resultant of the material stresses are in equilibrium with the given cross section force. The cross sections in the presented approach are made up of components of different materials. The approach permits modeled material stress-strain curves to have any reasonable shape - they are not restricted to linear elastic behaviour.

The sign conventions used in this paper are tension positive and compression negative for stresses, strains, and axial forces. A positive bending moment and curvature is one that would produce tension on the bottom of the cross section.

2.2 Reinforcing Material Behaviour

Although mild reinforcing materials normally exhibit a post-yield strengthening associated with strain hardening, it is common to model the stress-strain curve for mild reinforcement as linear elastic behaviour to the yield stress with fully plastic behaviour beyond. It is also possible to apply partial material factors as required by some codes and standards directly to the reinforcing stress-strain curves.

Prestressing materials do not typically experience a clearly defined yield point. In prestressing materials the yield stress is generally determined by a reference strain. The modeled stress-strain curves for prestressing can assume any reasonable shape that closely predicts the real material behaviour. An example of such a curve was presented by Develapura and Tadros (2). Additionally, prestress force is sometimes considered as an external load for some purposes and an internal force for others. It is important to keep the treatment consistent in the cross section curvature calculations. For service calculations, it is common to treat the prestressing force from the initial prestress as an external load, in which case they become part of the cross section design forces. It is then only necessary to consider changes in the tendon stress as internal forces in the curvature calculations. The externally applied loads can cause a positive or negative change to the tendon stress.

Bonded prestressing materials normally undergo significant strains before they are bonded to the concrete. These “prestrains” are important to consider in the curvature calculations. Normally the cross section strain at the tendon elevation is modified by this prestrain to find a prestress component strain that can be used to determine the tendon stress from the stress-strain curve.

In the case of unbonded post-tensioned tendons, the prestressing material is never bonded to the concrete. In this case, isolated strains that occur at the crack locations are generally spread over a length of the tendon. This causes the stress in unbonded tendons at crack locations to increase at a much lower rate than bonded reinforcement. The effective component strain on unbonded tendons is normally predicted using semi-empirical equations in regional codes and standards.

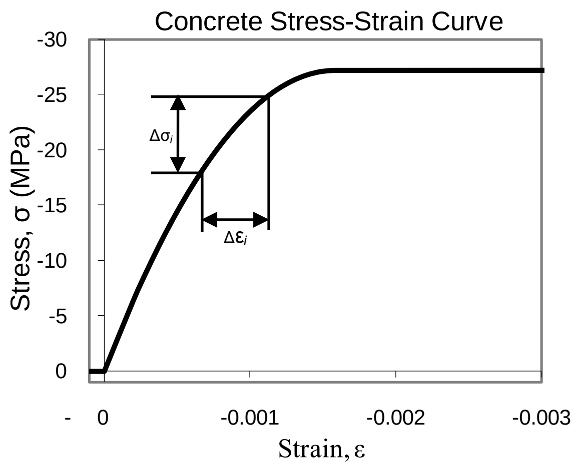
Other reinforcing materials can also be used in the cross sections. All that is generally required is a material stress-strain curve that appropriately models the material’s behaviour. Some examples are prestressing bars, carbon fibre strips or fabric, or glass fibre tendons. In using any material, it is important to consider any prestrains that might exist on such materials in the determination of the component material strains used to determine the stress from the stress-strain curves..

2.3 Concrete Material Behaviour

For modeling concrete behaviour, any reasonable stress-curve can be adopted. It is common to assume a parabolic curve up to a specified transition strain with constant stress under increasing strains (see Fig. 2.1). Concrete has several unique characteristics that significantly affect the long-term deflection behaviour of reinforced or prestressed concrete members. These factors include creep, shrinkage, cracking, and load history effects. Each of these effects is discussed in more detail here.

2.3.1 Creep

Under sustained load, concrete strains increase with time due to the phenomenon normally referred to as creep. Creep strains occur over time and a number of models are available to predict the percentage of total creep as a function of time. The creep factor normally represents a linear factor representing a multiple of the initial load induced concrete strain. There are a number of methods available to predict the creep coefficient, although normal creep coefficients fall within the range of 2.0 – 2.5. In the formulation in this paper, a single creep coefficient is used for concrete in compression as well as in tension although Atrushi (3) reported that creep values for concrete in tension are slightly higher than those in compression. It will be shown later that the cross section calculations rely heavily on the assumption that concrete does not carry any tension and so this assumption does not significantly influence the results. In order to consider loads that are applied at different times, it is also necessary to assume that creep strains of like or opposing signs can be superimposed. This assumption is also likely to be very reasonable for the normal range of service loads.



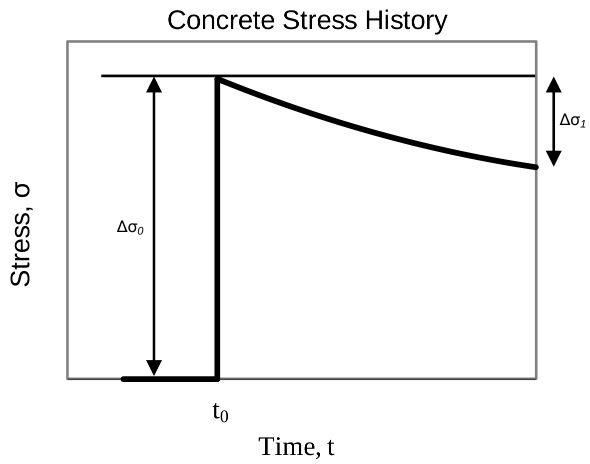


Figure 2.1. Concrete Stress-Strain Curve

Figure 2.2. Non-uniform Loading Diagram

For an unrestrained specimen that is subjected to an instantaneously applied load causing a constant stress and an initial strain ε_0 applied at time t_0 and sustained until time t, the total long term concrete strain is:

$$\varepsilon_{LT}(t, t_0) = \varepsilon_0[1 + \varphi(t, t_0)] \qquad (2\text{-}1)$$

where the nomenclature (t,t_i) is taken as the effect for the time period beginning at time t_i and ending at time t. The symbol $\varphi(t, t_0)$ thus represents the portion of the total creep coefficient φ that occurs between time t_0 and t. The concrete stress σ_0 and short term strain ε_0 are related by the concrete stress-strain curve. Given one, the other can be determined from the curve.

If instead the load $\Delta\sigma$ is applied gradually over a period of time from t_0 to t, the total long term concrete strain can be represented as:

$$\varepsilon_{LT}(t, t_0) = \varepsilon_0[1 + \chi\varphi(t, t_0)] \qquad (2\text{-}2)$$

where the factor $\chi(t, t_0)$ is commonly referred to as the ageing coefficient and is used to describe the rate of application of the loading, its effect on the total creep, and the variation of concrete strength over the time period. While its rigorous calculation is rather involved, this value can for most typical problems be assumed to be 0.8 with little loss in accuracy. A derivation and thorough discussion of this coefficient is presented by Ghali et al. (1). The symbol $\chi\varphi(t, t_0)$ represents the product of the terms $\varphi(t, t_0)$ and $\chi(t, t_0)$.

In Figure 2.2 there is a combination of instantaneous and gradually applied loads. In this case the total strain is a sum of equations 2-1 and 2-2:

$$\varepsilon_{LT}(t,t_0) = \Delta\varepsilon_0[1+\varphi(t,t_0)] + \Delta\varepsilon_1[1+\chi\varphi(t,t_0)] \qquad (2\text{-}3)$$

Where $\Delta\varepsilon_0 = f(\Delta\sigma_0)$ and $\Delta\varepsilon_1 = f(\Delta\sigma_1)$ from the concrete stress-strain curves. For the general case with an unlimited number of arbitrary load/time steps, the total long term strain including creep can be represented as:

$$\varepsilon_{LT}(t,t_0) = \sum_{i=0}^{n} \Delta\varepsilon_i[1+\chi\varphi(t,t_i)] \qquad (2\text{-}4)$$

where χ is taken as 1.0 for strains resulting from instantaneously applied loads and 0.8 for strains resulting from gradually applied loads. $\Delta\varepsilon_i$ is the strain change that would result from an instantaneous change in stress $\Delta\sigma_i$ at the beginning of the time period and is calculated from the concrete stress-strain curve as the difference from the end strain of the previous step.

2.3.2 Shrinkage

Shrinkage is the reduction in concrete volume over time due to hydration of cement, loss of moisture, and other factors. It occurs independent of loading and is normally specified as a strain. The normal range is 0.0004 to 0.0008.

The restraint of shrinkage movements result in stresses being imposed on the materials in the cross section. Restraint from external restraining elements (stiff columns/walls, adjacent slab areas, etc.) can cause a gradual buildup of tensile stress in the concrete. This tensile stress can serve to decrease the load required to cause cracking of the section (or even cause cracking in itself), and thus change the cracking response of the member.

Restraint to shortening can also result from eccentrically placed bonded reinforcement in the cross section. As the member tries to shorten, this reinforcement will restrain the movement on one face causing curvature on the cross section. Because the reinforcement is normally on the slab face that is in tension due to applied loads, the curvature due to shrinkage warping tends to amplify the curvature due to applied loads, thus increasing deflections.

2.3.3 Cracking

While concrete has some ability to carry tensile stresses, the flexural tensile strength is approximately 1/10 of the compression capacity. When a flexural load causes the applied tensile stresses to exceed the cracking stress, the stress is relieved at that location and a redistribution of stress occurs with a resulting increase in cross section curvature. As load increases, the number of cracks generally also increases. In the cross section calculations, at the crack locations the concrete is assumed to carry no tension. In the regions between the cracks the bonded tension reinforcement transfers tension back into the concrete. This behaviour is normally referred to as tension stiffening. In a partially cracked member, the mean curvature over a region lies between the uncracked and cracked response. One of the original tension stiffening models was developed by Branson (4). However, due to the realization that Branson's model overestimates the stiffness in members with low reinforcing ratios, other models have since been developed to better predict deflections in members with low reinforcing ratios such as slabs. One such model is presented in the Eurocode 2-2004 (5):

$$\alpha = \zeta\alpha_{11} + (1-\zeta)\alpha_1 \qquad (2\text{-}5)$$

where α is the mean curvature, α_1 is the uncracked curvature, and α_{11} is the fully cracked curvature and ζ is a distribution coefficient:

$$\zeta = 1 - \beta\left(\frac{\sigma_{sr}}{\sigma_s}\right)^2 \qquad (2\text{-}6)$$

and β is taken as 1.0 for instantaneous loads and 0.5 for sustained loads, σ_s is the stress in the tension reinforcement in the cracked section, and σ_{sr} is the stress in the tension reinforcement under the loading conditions causing first cracking. This model has shown good correlation with test results for a wide range of reinforcing ratios. There is also a factor to account for the quality of bond with the reinforcement which is not presented here for simplicity.

Tests conducted by Scott and Beeby (6) show that over time the tension in the concrete between the cracks reduces to about half its initial value. In their tests, this loss of tension stiffening occurred very rapidly. The reason for the reduction in stress is likely related to tensile creep and increased cracking over time, as well as possibly degradation of bond of the reinforcement near the cracks. This is the reason for the application of the β factor for long-term loads in the Eurocode 2 equation.

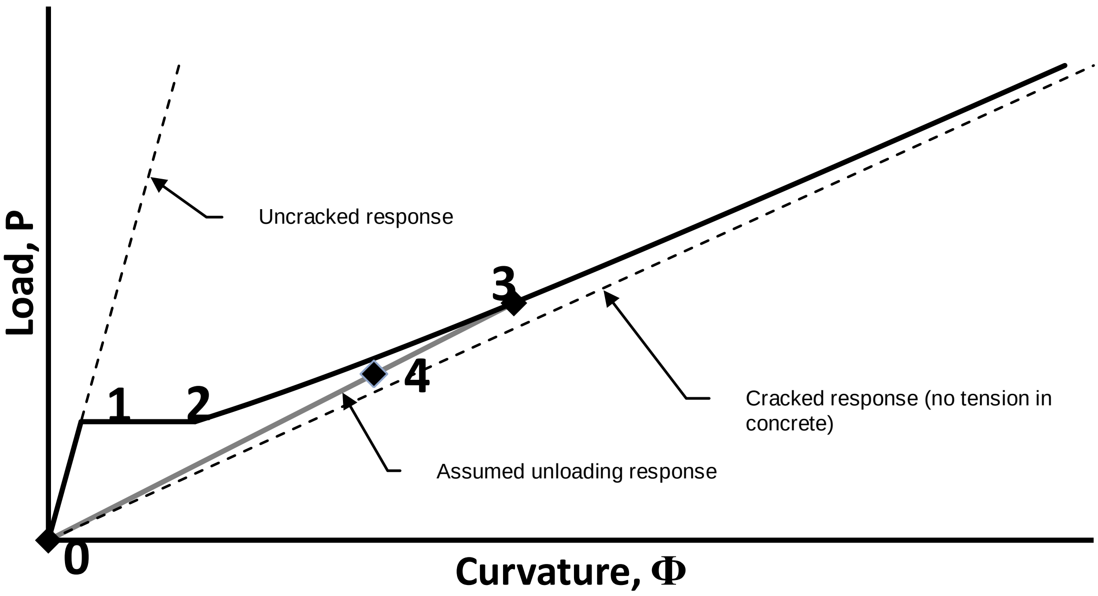


Figure 2.3. Tension Stiffening Effects for Loading/Unloading

The tension stiffening model is qualitatively plotted in Figure 2.3 with a β of 0.5. The cracking load is reached at point 1. As load increases post-cracking between points 2 and 3, the mean response approaches the cracked response. Strictly speaking, the cracked and uncracked response will only be linear for linear elastic material behaviour. Additionally, creep strains may cause the structure to unload along a different line than the loading curve, which would result in an offset strain. Despite these complications, it is reasonable to apply the tension stiffening model to these cases since it simply interpolates between cracked and uncracked states.

2.3.4 Load and Time History Effects

While the tension stiffening model predicts the cracked response for instantaneous loads, and adequately accounts for the degradation of tension stiffening over time, it does not adequately handle other issues related to the load history effects on the member.

If the member is loaded beyond cracking at an early age, then calculating the creep response based upon a fully cracked member is reasonable. However, if the cracking load is not reached for some time after application of sustained loads then considering creep effects as if the member were cracked for the entire duration could greatly overestimate the creep. While calculating the cracked section response, the cracked cross section curvature should always be used to determine instantaneous strains, but the creep strains considered should take into account whether or not the cross section has actually cracked. Also, when calculating the cracked state (where the concrete stress-strain curve does not consider tension) creep should only be applied to portions of the strain change in compression.

The tension stiffening model is frequently used with $\frac{M_{cr}}{M_a}$ in place of $\frac{\sigma_{sr}}{\sigma_s}$ in the Eurocode equation. For members without significant axial loads and assumed linear elastic material behaviour, this representation is theoretically equivalent. For these cases the cracking moment can be modified to account for the effect of shrinkage restraint. One formula to account for the restraining effects of reinforcement was proposed by Gilbert (7) and adapted by Scanlon and Bischoff (8). When axial loads are present, as is the case in prestressed members these loads must be appropriately accounted for.

Another issue with the tension stiffening model is that it does not generally account for unloading, so if the sustained load is much less than the peak load the model is not directly applicable. In order to account for the long-term effects some extensions are made to the model. When a cross section has been loaded such that cracking has occurred, when the loading is reduced the cracks will tend to close. However, due to slip at the crack location caused by the shear on the cross section the irregularities at the crack no longer align and the crack cannot close completely. Despite this fact, it seems reasonable to assume that the cracks can close completely and the unloading takes place in a linear fashion as shown in Figure 2.3 along line 3-0. It is expected that the error due to this effect will be small except for very low loading levels when cracking and deflections are unlikely to be critical.

To follow the entire load history for a typical cross section, loading increases along line 0-1 until cracking occurs. The long-term β factor is applied at this point, bringing the curvature to point 2 with no further increase in load. Further loading increases the curvature until the cross section reaches its maximum loading at point 3. After this, the specimen is unloaded to point 4, where the load is sustained for a long period of time. This is the final average curvature for the cross section. It is important to note that the cracked state response inherently considers the cracking history of the cross section when determining creep contributions. If the cracking occurs early in the load history, the slope of the curve will be shallower than if the cracking occurs late in the load history.

2.3.5 A Comprehensive Solution

The presented solution is an adaptation of the approach developed by Ghali et al. (1) and takes into account all the previously mentioned factors. Given any arbitrary load history of cross section forces, the complete stress and strain history on the cross section can be determined. Equation 2-1 can be used to determine the effects of rapid change in force at the beginning of a time interval, and equation 2-2 can be used to account for the gradual creep and redistribution of forces due to shrinkage strains occurring over the time period. While the resultant cross section force can change at the beginning of a time period (applied as an instantaneous load), the assumption is made that the resultant cross section force remains the same over the remainder of the time period, and is thus treated as a sustained load. The complete history is determined by solving for the stress and strain differentials in each time period sequentially. The entire stress and strain history can be determined by summing the stress and strain changes in each time period as shown.

$$\varepsilon_{LT}(t,t_0) = \sum_{i=0}^{n} \Delta\varepsilon_i[1+\chi\varphi(t,t_i)] + \varepsilon_{sh}(t,t_0) \qquad \text{(2-7)}$$

where the factor $\chi(t,t_0)$ is taken as 1.0 for instantaneous strain changes and is assumed to be 0.8 for strain changes occurring gradually over a time period. In rearranging this equation the instantaneous strain change in any time period can be solved if the previous time periods are already known.

$$\Delta\varepsilon_j = \frac{1}{1+\chi\varphi(t,t_j)}\left\{\varepsilon_{LT}(t,t_0) - \varepsilon_{sh}(t,t_0) - \sum_{i=0}^{j-1} \Delta\varepsilon_i[1+\chi\varphi(t,t_i)]\right\} \qquad \text{(2-8)}$$

Using this equation, and iteratively assuming a value for the long term strain ε_{LT}, all other values are known and the resulting instantaneous strain change in a time period can be determined. Subsequently, the strain change in a time period can be used to determine the stress change in a period from the stress-strain curve. When determining the stress change from the stress-strain curve, it is important to take into consideration the cumulative strain changes from the previous time periods. By solving for the strain change in each time period successively the stress history for the entire load history can be determined. The stress at time t is simply the sum of the increments:

$$\sigma(t,t_0) = \sum_{i=0}^{n} \Delta\sigma_i \qquad \text{(2-9)}$$

Using this technique, the cross section resultant force can be solved for by iteratively assuming a cross section axial strain and curvature, using equations 2-8 and 2-9 to solve for the stress in the concrete at each fibre, and performing numerical integration over the concrete stress blocks to solve for the resultant concrete force. By iteratively assuming long-term strains ε_{LT} and solving for the resultant concrete stresses and calculating the resultant cross section force, a final long term strain distribution that results in the resultant cross section force equivalent to the desired internal force is determined.

Even after performing accurate cross section strain calculations at discrete locations, the task of applying those results to the global analysis in the continuum exists. One such approach to this problem is presented in section 3.

3. Modified Stiffness Approach

Once the cross section curvatures have been determined with reasonable accuracy in accordance with section 2, the results can be applied back to the global analysis. The calculated long-term cross section deformations are not consistent with the respective deformations in the global analysis because the global stiffness analysis is generally based upon gross section properties without consideration of long-term effects, while the cross section deformations consider cracking along with the noted long-term effects.

In order to resolve the curvature inconsistencies the stiffness of the elements in the finite element model can be adjusted (many software packages allow stiffness factors to be specified in orthogonal directions). A stiffness adjustment factor can be calculated as:

$$k_i = \frac{\Phi_{gross}}{\Phi_{LT}}$$

where k is a stiffness factor for a particular axis and Φ_{gross} is the gross cross section curvature and Φ_{LT} is the mean cross section curvature calculated considering cracking and long-term effects. Modifying the stiffness in the elements used in the global analysis forces the curvatures and thus deflections in the linear elastic global analysis to be consistent with the results from the detailed cross section calculations. The stiffness factors are applied to the orthogonal element properties based upon the orthogonally placed cross sections and their corresponding curvatures. The torsion properties of the elements should be modified by an average factor, and for slabs with changes in centroid elevations it may become important to modify the axial stiffness of the slabs as well to properly account for the bending stiffness component represented by the eccentric axial element stiffness. While the presented cross section solutions can also result in axial strains in addition to curvatures, adjustment of element axial stiffness to account for these strains is generally not important as for most structures the concentric axial loads and deformations will have little effect on the out of plane deflection of the global analysis of the structure.

The modification of the element stiffness will normally cause a redistribution of forces within the structure (as cracking and long-term deformations will tend to do in the real structure). Due to this effect, iterations will be necessary until the cross section forces converge, at which point the solution has been reached.

4. Conclusions

A consistent approach has been presented that takes into account all factors considered to be important in the calculation of long-term deflections of structures. The approach also takes into consideration the time and load history of the member and the interrelation of the different effects. The approach consists of modifying the concrete material strains for the long-term effects before using the modified strains to calculate stress changes for each interval from the concrete stress-strain curve. Once the stress changes in each interval are summed to find the stress history for a concrete fibre, the stresses at numerous discrete points are calculated and cross section resultants determined. The cross section curvatures and axial strains are adjusted iteratively until the calculated cross section resultants are equivalent to the cross section force calculated from the global analysis. The approach is generic and applies well to structures containing a variety of reinforcement types, including prestressing.

Using this technique in a flat plate or other continuum structure consists of defining a grid of orthogonal cross sections. Once cross section curvatures are calculated for the long-term effects, stiffness factors are calculated and applied to the linear elastic global finite element analysis. This allows contour plots to be calculated for the cracked long-term effects and allows for quick and easy evaluation of the long-term deflections for applicable code or project criterion.

Because the method requires an iterative approach, it is suitable for implementation in a computer program.

5. Notation

$\varepsilon_{LT}(t,t_0)$	=	total concrete strain including long term effects occurring from time t_0 to t
$\varepsilon_{sh}(t,t_0)$	=	free shrinkage strain occurring from time t_0 to t
$\Delta\varepsilon_i$	=	change in strain induced in the concrete in time period i
$\Delta\sigma_i$	=	change in concrete stress in time period i
$\varphi(t,t_0)$	=	creep coefficient effective for time period t_0 to t
Φ	=	curvature on a cross section (1/mm)
$\chi(t,t_0)$	=	ageing coefficient for time period t_0 to t
$\chi\varphi(t,t_0)$	=	product of creep coefficient and ageing coefficient for time period t_0 to t

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Appendix: Worked Examples

Example 1: concentrically loaded cross section

A variable concentric axial force is applied to the reinforced concrete cross section shown in Figure A1. Calculate the concrete and reinforcement stresses at each stage given the following data: E_c = 25.0 GPa; Es = 200 GPa; Total free shrinkage = 0.0006; Moist curing ends at 7 days; Total creep coefficient = 2.5; The concrete and reinforcing materials are assumed to be linear elastic. The concrete is assumed to remain uncracked. The cross section strains in the problem are determined by iteration to determine component stresses in equilibrium with the external loads. The load history is as follows:

Stage 1: t = 0 to 30 days, no loads applied
Stage 2: t = 30 to 100 days, P = 100 kN (tension)
Stage 3: t = 100 to 10000 days, P = -2000 kN (compression)

(a) Stage 1, t = 0 to 30 days
During this stage, no loads are applied to the structure. However, shrinkage will occur and will be restrained by the reinforcement, causing stresses on both components. The fraction of the total shrinkage occurring during this time period is 39.6%, or a shrinkage strain of -237.9 x 10^{-6}. The creep factor occurring during this period is φ(30,7) = 0.984. Because the stresses are induced gradually, use χ=0.8.

ε_{LT} = -219 x 10^{-6}

ε_s = -219 x 10^{-6} ; σ_s = -44 MPa

For the effective concrete strain change using equation 2-8,

$$\Delta\varepsilon_{c1} = \frac{1}{1+0.8(0.984)}\{-219 \text{ x } 10^{-6} - -237.9 \text{ x } 10^{-6}\} = 1.05 \times 10^{-5}$$

And then using equation 2-9 to determine the concrete stress change,

$$\Delta\sigma_{c1} = (1.05 \times 10^{-5})25000 = 0.26\ MPa$$

(b) Stage 2, t = 30 to 100 days

At the beginning of this time period a tension of 100 kN is applied to the cross section. The cross section strain after applying this load is -215.3 x 10^{-6}. This results in a reinforcement stress of -43 MPa. The effective concrete strain change is:

$$\Delta\varepsilon_{c2,instant} = \frac{1}{1+1.0(0.0)}\{-215.3 \text{ x } 10^{-6} - -237.9 \text{ x } 10^{-6} - 1.05 \times 10^{-5}(1+0.8(0.984))\} = 3.82 \times 10^{-6}$$

And the concrete stress change,

$$\Delta\sigma_{c2,instant} = (3.82 \times 10^{-6})25000 = 0.096\ MPa$$

For the remainder of the time period, the cross section force remains at 100 kN and the creep and shrinkage will be considered. The fraction of the total shrinkage occurring at the end of this time period is 72.7%, or a shrinkage strain of -435.9 x 10^{-6}. The creep factors used during this period are φ(100,30) = 1.174 and φ(100,7) = 1.497. The cross section strain at the end of this stage is -388.4 x 10^{-6} resulting in a reinforcement stress of -78 MPa. The effective concrete strain change is:

$$\Delta\varepsilon_{c2,sustained} = \frac{1}{1+0.8(1.174)}\{-388.4 \text{ x } 10^{-6} - -435.9 \text{ x } 10^{-6} - 1.05 \times 10^{-5}(1+0.8(1.497)) - 3.82 \times 10^{-6}(1+1.0(1.174))\} = 8.31 \times 10^{-6}$$

And the concrete stress change,

$$\Delta\sigma_{c2,sustained} = (8.31 \times 10^{-6})25000 = 0.208\ MPa$$

(b) Stage 3, t = 100 to 10000 days

At the beginning of this time period a compression of 2000 kN is applied to the cross section. The cross section strain after applying this load is -468.5 x 10^{-6}. This results in a reinforcement stress of -94 MPa. The effective concrete strain change is:

$$\Delta\varepsilon_{c3,instant} = \frac{1}{1+1.0(0.0)}\{-468.5 \times 10^{-6} - -435.9 \times 10^{-6} - 1.05 \times 10^{-5}(1+0.8(1.497)) - 3.82 \times 10^{-6}(1+1.0(1.174)) - 8.31 \times 10^{-6}(1+0.8(1.174))\} = -8.01 \times 10^{-5}$$

And the concrete stress change,

$$\Delta\sigma_{c3,instant} = (-8.01 \times 10^{-5})25000 = -2.0\ MPa$$

For the remainder of the time period, the cross section force remains at -2000 kN and the creep and shrinkage will be considered. The fraction of the total shrinkage occurring at the end of this time period is 99.7%, or a shrinkage strain of -597.9 x 10^{-6}. The creep factors used during this period are φ(10000,100) = 1.745; φ(10000,30) = 2.012 and φ(10000,7) = 2.389. The cross section strain at the end of this stage is -724.7 x 10^{-6} resulting in a reinforcement stress of -145 MPa. The effective concrete strain change is:

$$\Delta\varepsilon_{c3,sustained} = \frac{1}{1+0.8(1.745)}\{-724.7 \times 10^{-6} - -597.9 \times 10^{-6} - 1.05 \times 10^{-5}(1+0.8(2.389)) - 3.82 \times 10^{-6}(1+1.0(2.012)) - 8.31 \times 10^{-6}(1+0.8(2.012)) - -8.01 \times 10^{-5}(1+1.0(1.745))\} = 1.229 \times 10^{-5}$$

And the concrete stress change,

$$\Delta\sigma_{c3,sustained} = (1.229 \times 10^{-5})25000 = .307\ MPa$$

Time (days)	Load P (kN)	ε_{LT} (x 10^{-6})	σ_s (MPa)	$\Delta\varepsilon_c$ (x 10^{-6})	$\Delta\sigma_c$ (MPa)	σ_c (MPa)
30	0	-219	-44	10.5	0.26	0.26
30	100	-215.3	-43	3.82	0.096	0.36
100	100	-388.4	-78	8.31	0.208	0.56
100	-2000	-468.5	-94	-80.1	-2.0	-1.44
10000	-2000	-724.7	-145	12.29	0.307	-1.13

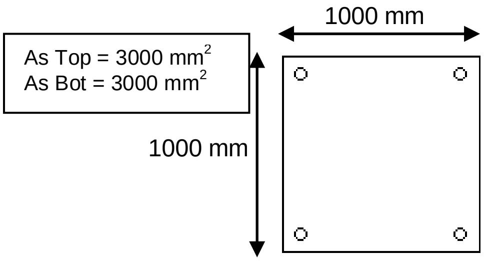

Figure A1. Example 1

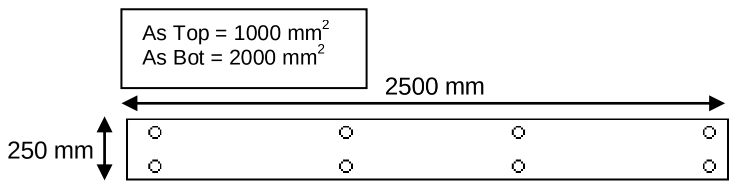

Figure A2. Example 2

Example 2: eccentrically loaded cross section

A variable moment and axial force combination is applied to the reinforced concrete cross section shown in Figure A2. Calculate the axial strain and curvature at each stage given the following data: f_c' = 32.0 MPa; Es = 200 GPa; Total free shrinkage = 0.0006; Moist curing ends at 7 days; Total creep coefficient = 2.5; The reinforcing materials are linear elastic-plastic, the concrete uses a parabolic/constant curve similar to Figure 2.1. The cross section curvatures and axial strains in the problem are determined by iteration to determine component stresses in equilibrium with the external loads. The load history is as follows:

Stage 1: t = 3 to 90 days, construction load of P = -100 kN, M = 85 kN-m
Stage 2: t = 90 to 10000 days, sustained service load of P = -100 kN, M = 65 kN-m

Consider the effect of instantly applying and then removing a cracking load = 1.5 Pcr at times t = 3 days, t = 90 days, and t = 10000 days. The cross section is assumed to remain uncracked up to the time of application of this load. In the last column the curvatures are normalized to the curvature at 10000 days for the case where the cracking load is applied at 3 days.

Cracking load (days)	Time (days)	Load (kN, kN-m)	Uncracked Curvature (1/mm x 10^{-8})	Cracked Curvature (1/mm x 10^{-8})	Mean Curvature (1/mm x 10^{-8})	Bottom Steel Stress (MPa)	Top Fibre Concrete Stress (MPa)	Normalized Mean Curvature
3	3	-100, 85	91.3	514.0	419.7	179.0	-6.65	0.632
	90	-100, 85	237.6	803.7	677.5	183.2	-4.35	1.02
	90	-100, 65	216.1	671.2	569.7	136.4	-2.73	0.858
	10000	-100, 65	269.4	771.1	663.9	135.3	-2.51	1.0
90	3	-100, 85	91.3	-	91.3	16.8	-3.2	0.137
	90	-100, 85	237.6	774.3	654.6	178.3	-6.34	0.986
	90	-100, 65	216.1	640.2	545.6	130.6	-4.81	0.822
	10000	-100, 65	269.4	759.7	650.4	133.3	-3.03	0.98
10000	3	-100, 85	91.3	-	91.3	16.8	-3.2	0.137
	90	-100, 85	237.6	-	237.6	-35.0	-2.84	0.358
	90	-100, 65	216.1	-	216.1	-39.2	-2.13	0.326
	10000	-100, 65	269.4	740.5	635.4	130.9	-4.63	0.957